Higher Algebra with Operads

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Plan

1. Toy models

2. Operadic homotopical algebra

3. Homotopy Batalin–Vilkovisky algebras
Homotopy data and mixed complex structure

- **Homotopy data:** Deformation retract of chain complexes

  \[ h \xleftarrow{p} (A, d_A) \xrightarrow{i} (H, d_H) \quad \text{and} \quad \text{id}_A - ip = d_A h + hd_A. \]

- **Algebraic data:** \( \Delta : A \to A, \quad d_A \Delta + \Delta d_A = 0, \quad \Delta^2 = 0 \)

  mixed complex \( |\Delta| = 1 \) (or bicomplex).

- **Transferred structure:** \( \delta_1 := p \Delta i \)

  Does \( \delta_1 \) square to zero?

  \[(\delta_1)^2 = p \Delta \underbrace{ip \Delta i}_{\sim_h \text{id}_A} \neq 0 \text{ in general!} \]

  Idea: Introduce \( \delta_2 := p \Delta h \Delta i \)

  Then, \( \partial(\delta_2) = (\delta_1)^2 \) in \((\text{Hom}(A, A), \partial := [d_A, -])\).

  \( \Rightarrow \delta_2 \) is a homotopy for the relation \((\delta_1)^2 = 0\).
Higher structure: multicomplex

Higher up, we consider: 

\[ \delta_n := p(\Delta h)^{n-1} \Delta i, \quad \text{for } n \geq 1. \]

**Proposition**

\[ \partial(\delta_n) = \sum_{k=1}^{n-1} \delta_k \delta_{n-k} \quad \text{in } (\text{Hom}(A, A), \partial), \quad \text{for } n \geq 1. \]

**Definition (Multicomplex)**

\( (H, \delta_0 := -d_H, \delta_1, \delta_2, \ldots) \) graded vector space \( H \) endowed with a family of linear operators of degree \( |\delta_n| = 2n - 1 \) satisfying

\[ \sum_{k=0}^{n} \delta_k \delta_{n-k} = 0, \quad \text{for } n \geq 0. \]

**Remark:** A mixed complex = multicomplex s.t. \( \delta_n = 0, \) for \( n \geq 2. \)
Multicomplexes are homotopy stable

- Starting now from a multicomplex \((A, \Delta_0 = -d_A, \Delta_1, \Delta_2, \ldots)\)
- Consider the transferred operators

\[
\delta_n := \sum_{k_1 + \cdots + k_l = n} p \Delta_{k_1} h \Delta_{k_2} h \cdots h \Delta_{k_l}, \quad \text{for } n \geq 1
\]

**Proposition**

\[
\partial(\delta_n) = \sum_{k=1}^{n-1} \delta_k \delta_{n-k} \quad \text{in } (\text{Hom}(A, A), \partial), \quad \text{for } n \geq 1.
\]

\(\implies\) Again a multicomplex, no need of further higher structure.
Compatibility between Original and Transferred structures

\[(A, \Delta_0 = -d_A, \Delta_1, \Delta_2, \ldots) \xleftarrow{i} (H, \delta_0 = -d_H, \delta_1, \delta_2, \ldots)\]

Original structure

Transferred structure

- **i** chain map \(\implies \Delta_0 i = i \delta_0\)

**Question:** Does \(i\) commute with the \(\Delta\)'s and the \(\delta\)'s?

\[i \delta_1 = i p \Delta_1 i \neq \Delta_1 i \text{ in general!}\]

\[\sim_h \text{id}_A\]

- Define \(i_0 := i\) and consider \(i_1 := h \Delta_1 i\).

Then, \(\partial(i_1) = \Delta_1 i_0 - i_0 \delta_1\) in \(\text{Hom}(H, A), \partial\).

\(\implies i_1\) is a homotopy for the relation \(\Delta_1 i_0 = i_0 \delta_1\).
Higher up, we consider:

\[ i_n := \sum_{k_1+\cdots+k_l = n} h\Delta_{k_1} h\Delta_{k_2} h \cdots h\Delta_{k_l} i, \quad \text{for } n \geq 1. \]

⇒ \[ \partial(i_n) = \sum_{k=0}^{n-1} \Delta_{n-k} i_k - \sum_{k=0}^{n-1} i_k \delta_{n-k} \quad \text{in } (\text{Hom}(H, A), \partial), \quad \text{for } n \geq 1. \]

**Definition (\( \infty \)-morphism)**

\[ i_\infty : (H, \delta_0 = -d_H, \delta_1, \delta_2, \ldots) \rightsquigarrow (A, \Delta_0 = -d_A, \Delta_1, \Delta_2, \ldots) \]

collection of maps \( \{i_n : H \to A\}_{n \geq 0} \) satisfying

\[ \sum_{k=0}^{n} \Delta_{n-k} i_k = \sum_{k=0}^{n} i_k \delta_{n-k}, \quad \text{for } n \geq 0. \]
The category $\infty$-multicomp

**Proposition (Composite of $\infty$-morphisms)**

Let $f : A \rightsquigarrow B$, $g : B \rightsquigarrow C$ be two $\infty$-morphisms of multicompleses. Then

$$(gf)_n := \sum_{k=0}^{n} g_{n-k}f_k,$$

for $n \geq 0$,

defines an associative and unital composite of $\infty$-morphisms.

**Category:** multicomplesx with $\infty$-morphisms: $\infty$-multicomp.

**Compact reformulation:**

Multicomples = square-zero element

$$\Delta(z) = \Delta_0 + \Delta_1 z + \Delta_2 z^2 + \cdots$$

in the algebra $\text{End}_A[[z]]$,

$\infty$-morphisms = $i(z) \in \text{Hom}(H, A)[[z]]$ s.t. $i(z)\delta(z) = \Delta(z) i(z)$,

composite = $g(z)f(z)$.
Homotopy Transfer Theorem for multicomplexes

∞-quasi-isomorphism: \( i : H \xrightarrow{\sim} A \) s.t. \( i_0 : H \xrightarrow{\sim} A \) qi.

Theorem (HTT for multicomplexes, Lapin ’01)

Given any deformation retract

\[
\begin{align*}
(\mathcal{A}, d_{\mathcal{A}}) & \xrightarrow{h} (\mathcal{H}, d_{\mathcal{H}}) \\
\xRightarrow{i} & \quad \xLeftarrow{p}
\end{align*}
\]

\[\text{id}_{\mathcal{A}} - ip = d_{\mathcal{A}} h + hd_{\mathcal{A}}\]

and any multicomplex structure on \( A \), there exists a multicomplex structure on \( H \) such that \( i \) extends to an \( \infty \)-quasi-isomorphism.

Application 1: \( \mathbb{K} \) field, \( (\mathcal{A}, d, \Delta) \) bicomplex, \( (H(A, d), 0) = E^1 \) deformation retract

\( \Rightarrow \) multicomplex structure on \( H(A, d) \) = lift of the spectral sequence, i.e. \( \delta^r \Rightarrow d^r \).

Application 2: Equivalence between the various definitions of cyclic homology [Loday-Quillen, Kassel].
Homotopy theory of mixed complexes

**Definition (Homotopy category)**

Localisation with respect to quasi-isomorphisms

\[
\text{Ho}(\text{mixed cx}) := \text{mixed cx}[qi^{-1}]
\]

\[
\text{Hom}_{\text{Ho}}(A, B) := \{ A \to \bullet \leadsto \bullet \to \bullet \cdots \bullet \leadsto \bullet \to B \}/\sim
\]

**Theorem (?)**

*Every \(\infty\)-qi of multicomplexes admits a homotopy inverse.*

\[
\text{Ho}(\text{mixed cx}) \cong \infty\text{-mixed cx}/\sim_h.
\]

**Proof.**

[...]

*Rectification:*

\[
\exists \ Rect : \infty\text{-multicomp} \to \text{mixed cx}, \text{ s.t. } H \sim Rect(H).
\]
**Initial structure:** an associative product on $A$

$$ \nu : A \otimes^2 \to A, \quad \text{s.t.} \quad \nu(\nu(a, b), c) = \nu(a, \nu(b, c)). $$

**Transferred structure:** the binary product on $H$

$$ \mu_2 := p\nu i \otimes^2 : H \otimes^2 \to H. $$
First homotopy for the associativity relation

- Is the transferred $\mu_2$ associative? Answer: in general, no!
- Introduce $\mu_3$:

\[
\begin{align*}
\text{In } \text{Hom}(A^{\otimes 3}, A), \text{ it satisfies}
\end{align*}
\]

$\Rightarrow \mu_3$ is a homotopy for the associativity relation of $\mu_2$. 

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Higher structure

Higher up, in $\text{Hom}(H^\otimes n, H)$, we consider:

$$\mu_n := \sum_{\text{PBT}_n} \pm$$

$\nu_n$ for $n \geq 1$.

Proposition

$$\partial \left( \begin{array}{c} 1 \ 2 \ \ldots \ n \end{array} \right) = \sum_{k+l=n+1 \atop 1 \leq j \leq k} \pm$$

$\partial$ is the differential.
**Definition (A_∞-algebra, Stasheff ’63)**

An $A_\infty$-algebra is a chain complex $(H, d_H, \mu_2, \mu_3, \ldots)$ endowed with a family of multilinear maps of degree $|\mu_n| = n - 2$ satisfying

\[
\partial \left( \begin{array}{c} 1 \ 2 \ \cdots \ n \\ \end{array} \right) = \sum_{k+l=n+1}^{1 \leq j \leq k} \pm \begin{array}{c} 1 \ \cdots \ j \ \cdots \ k \\ \end{array}
\]

**Remark:** A dga algebra = $A_\infty$-algebra s.t. $\mu_n = 0$, for $n \geq 3$. 
$A_\infty$-algebras are homotopy stable

- Starting from an $A_\infty$-algebra $(A, d_A, \nu_2, \nu_3, \ldots)$

- Consider

\[
\mu_n = \sum_{\text{PT}_n} \pm \]

Proposition

\[
\partial \left( \begin{array}{c}
1 \\
2 \\
\vdots \\
n
\end{array} \right) = \sum_{\substack{k+l=n+1 \\
1 \leq j \leq k}} \pm
\]

\[1 \ldots l \]

\[1 \ldots \]

\[j \ldots k \]

\[\implies \text{Again an } A_\infty \text{-algebra, no need of further higher structure.} \]
Compatibility between Original and Transferred structures

\[(A, d_A, \nu_2, \nu_3, \ldots) \xleftarrow{i} (H, d_H, \mu_2, \mu_3, \ldots)\]

- **i** chain map \( \implies d_A i = id_H 

**Question:** Does \( i \) commutes with the \( \nu \)'s and the \( \mu \)'s? 

**Answer:** not in general! 

Define \( i_1 := i \) and consider in \( \text{Hom}(H^\otimes n, A) \), for \( n \geq 2 \):

\[i_n := \sum_{\text{PT}_n} \pm \]

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$A_\infty$-morphism

**Definition ($A_\infty$-morphism)**

$(H, d_H, \{\mu_n\}_{n \geq 2}) \leadsto (A, d_A, \{\nu_n\}_{n \geq 2})$ is a collection of linear maps

$$\{f_n : H^\otimes n \to A\}_{n \geq 1}$$

of degree $|f_n| = n - 1$ satisfying

**Example:** The aforementioned $\{i_n : H^\otimes n \to A\}_{n \geq 1}$. 
Homotopy Transfer Theorem for $A_\infty$-algebras

$\infty$-quasi-isomorphism: $i : H \xrightarrow{\sim} A$ s.t. $i_0 : H \xrightarrow{\sim} A$ qi.

**Theorem (HTT for $A_\infty$-algebras, Kadeshvili '82)**

Given any deformation retract

$$h \xhookrightarrow{i} (A, d_A) \xleftrightarrow{p} (H, d_H)$$

$$\text{id}_A - ip = d_A h + hd_A$$

and any $A_\infty$-algebra structure on $A$, there exists an $A_\infty$-algebra structure on $H$ such that $i$ extends to an $\infty$-quasi-isomorphism.

**Application:** $A = (C_{\text{Sing}}^\bullet (X), \cup)$, transferred $A_\infty$-algebra on $H_{\text{Sing}}^\bullet (X) =$ lifting of the (higher) Massey products.
The category $\infty$-$A_\infty$-$\text{Alg}$

Compact reformulation:

$A_\infty$-algebra $=$ square-zero coderivation in the coalgebra $T^c(sA)$,

$A_\infty$-morphism $=$ morphism of dg coalgebras $T^c(sA) \rightarrow T^c(sB)$.

composite $[?]$ $=$ composite of morphisms of dg coalgebras.

Category: $A_\infty$-algebras with $\infty$-morphisms: $\infty$-$A_\infty$-$\text{Alg}$.
Theorem (Munkholm ’78, Lefèvre-Hasegawa ’03)

- Every $\infty$-qi of $A_\infty$-algebras admits a homotopy inverse.
- $\text{Ho}(\text{dga alg}) := \text{dga alg}[qi^{-1}] \cong \infty\text{-dga alg}/\sim_h$.

Proof. Use

\[
\text{dga alg} \xleftarrow{\Omega} \xrightarrow{\text{(conil) dga coalg}} \text{B}
\]

\[
\text{Rect}
\]

\[
\infty\text{-}A_\infty\text{-alg} \xleftarrow{\cong} \text{quasi-free dga coalg}
\]

+ [...] + Rectification:

\[\exists \text{ Rect} : \infty\text{-}A_\infty\text{-Alg} \rightarrow \text{dga alg}, \text{ s.t. } H \sim \text{ Rect}(H)\]
**Exercise**: Consider your favorite category of algebras “of type $\mathcal{P}$” (eg. Lie algebras, associative algebras+unary operator $\Delta$, etc.).

- Find the good notions of $\mathcal{P}_\infty$-algebras and $\infty$-morphisms.
- Fill the diagram

\[
\begin{array}{ccc}
\text{dg } \mathcal{P}\text{-alg} & \xrightarrow{??} & \text{??} \\
\text{Rect} & \swarrow & \nearrow \\ \xleftarrow{??} \\
\infty-\mathcal{P}_\infty\text{-alg} & \xleftarrow{??} & \text{??}
\end{array}
\]

- to prove the Homotopy Transfer Theorem
- and the equivalence of categories

\[
\text{Ho}(\text{dg } \mathcal{P}\text{-alg}) := \text{dg } \mathcal{P}\text{-alg}[qi^{-1}] \cong \infty-\text{dg } \mathcal{P}\text{-alg}/\sim_h.
\]
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Multilinear Operations: \( \text{End}_A(n) := \text{Hom}(A^\otimes n, A) \)

Composition:

\[
\text{End}_A(k) \otimes \text{End}_A(i_1) \otimes \cdots \otimes \text{End}_A(i_k) \rightarrow \text{End}_A(i_1 + \cdots + i_k)
\]

\[
g \otimes f_1 \otimes \cdots \otimes f_k \mapsto g(f_1, \ldots, f_k)
\]

Definition (Operad)

- Collection: \( \{ P(n) \}_{n \in \mathbb{N}} \) of \( S_n \)-modules
- Composition: \( P_k \otimes P_{i_1} \otimes \cdots \otimes P_{i_k} \rightarrow P_{i_1 + \cdots + i_k} \)
Examples of Operads

Definition (Algebra over an Operad)

Structure of \( P \)-algebra on \( A \): morphism of operads \( P \to \text{End}_A \)

Examples:

- \( D = T(\Delta)/(\Delta^2) \)-algebras (modules) = mixed complexes.
- \( \text{As} = T(\gamma) / (\gamma - \gamma) \)-algebras = associative algebras.

- Little discs \( D_2 \): \( D_2 \)-algebras \( \cong \) double loop spaces \( \Omega^2(X) \)
Example in Geometry

Deligne–Mumford moduli space of stable curves: $\overline{M}_{g,n+1}$

Definition (Frobenius manifold, aka Hypercommutative algebras)

Algebra over $H_\bullet(\overline{M}_{0,n+1})$, i.e. $H_\bullet(\overline{M}_{0,n+1}) \to \text{End}_{H_\bullet(A)} \iff$ totally symmetric $n$-ary operation $(x_1, \ldots, x_n)$ of degree $2(n - 2)$,

$$\sum_{S_1 \sqcup S_2 = \{1, \ldots, n\}} ((a, b, x_{S_1}), c, x_{S_2}) = \sum_{S_1 \sqcup S_2 = \{1, \ldots, n\}} \pm (a, (b, x_{S_1}, c), x_{S_2}).$$
Homotopy algebra and operads

\[
\text{operad } \mathcal{P} \xleftarrow{\sim} \mathcal{P}_\infty : \text{ quasi-free replacement (cofibrant)}
\]

category of \(\mathcal{P}\)-algebras \xleftarrow{} \text{category of homotopy } \mathcal{P}\text{-algebras}

Examples:

1. \(\mathcal{P} = D\) : \(D_\infty\)-algebras = multicomplexes

\[
D = \frac{T(\Delta)}{\Delta^2} \xleftarrow{\sim} D_\infty := \left(\frac{T(\delta \oplus \delta^2 \oplus \delta^3 \oplus \cdots), d_2}{\text{quasi-free}}\right).
\]

2. \(\mathcal{P} = \text{Ass}\) : \(\text{Ass}_{\infty}\)-algebras = \(A_{\infty}\)-algebras

\[
\text{As} = \frac{T(\bigcup)}{\left(\bigcup - \bigcup\right)} \xleftarrow{\sim} A_{\infty} := \left(\frac{T(\bigcup \oplus \bigcup \oplus \cdots), d_2}{\text{quasi-free}}\right).
\]
Koszul duality theory

\[ \mathcal{P}_\infty = \mathcal{T}(\text{operadic syzygies}) \xrightarrow{?\sim ?} \mathcal{P} \]

- **Quadratic presentation:** \( \mathcal{P} = \mathcal{T}(V)/(R) \), where

\[ R \subset \mathcal{T}^{(2)}(V) \]

trees with 2 vertices.

- **Koszul dual cooperad:** quadratic cooperad

\( \mathcal{P}^i := C(sV, s^2R) \), i.e. defined by a (dual) universal property.

- **Candidate:** \( \mathcal{P}_\infty = \Omega \mathcal{P}^i = \mathcal{T}(\mathcal{P}^i) \xrightarrow{?\sim ?} \mathcal{P} \).

- **Criterion:** Quasi-isomorphism iff the Koszul complex \( \mathcal{P} \circ_\kappa \mathcal{P}^i \) is acyclic.

- **Examples:** \( D, \text{Ass}, \text{Com}, \text{Lie} \), etc.
For any Koszul operad $\mathcal{P}$

- $\exists$ a notion of composable $\infty$-morphisms: $\infty$-$\mathcal{P}_\infty$-$\text{Alg}$.

$\mathcal{P}_\infty$-algebra $=$ square-zero coderivation in the coalgebra $\mathcal{P}^i(A)$, $\infty$-morphism $=$ morphism of dg coalgebras $\mathcal{P}^i(A) \to \mathcal{P}^i(B)$.

**Theorem (HTT for $P_\infty$-algebras, Galvez–Tonks-V.)**

*Given any deformation retract*

$$h \leftarrow (A, d_A) \xrightarrow{p} (H, d_H) \xleftarrow{i} \text{id}_A - ip = d_A h + hd_A$$

and any $\mathcal{P}_\infty$-algebra structure on $A$, there exists a $\mathcal{P}_\infty$-algebra structure on $H$ such that $i$ extends to an $\infty$-quasi-isomorphism.

Every $\infty$-qi of $\mathcal{P}_\infty$-algebras admits a homotopy inverse.

$\text{Ho}(\text{dg } \mathcal{P}\text{-alg}) := \text{dg } \mathcal{P}\text{-alg}[qi^{-1}] \cong \infty \text{-dg } \mathcal{P}\text{-alg}/\sim_h$.

**Proof.** Use

$\begin{align*}
\text{dg } \mathcal{P}\text{-alg} & \xrightarrow{\Omega_\kappa} (\text{conil}) \text{ dg } \mathcal{P}^i\text{-coalg} \\
\text{Rect} & \quad (\conil) \text{ dg } \mathcal{P}^i\text{-coalg} \\
\infty \text{-} \mathcal{P}_\infty\text{-alg} & \xleftarrow{\leftarrow} \text{ quasi-free } \mathcal{P}^i\text{-coalg}
\end{align*}$

+ Model Category on (conil) dg $\mathcal{P}^i$-coalg: we $\not\subseteq qi$

+ Rectification:

$\exists \text{ Rect: } \infty \text{-} \mathcal{P}_\infty\text{-Alg} \to \text{dg } \mathcal{P}\text{-alg}, \text{ s.t. } H \sim \text{ Rect}(H)$
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Batalin–Vilkovisky algebras

Definition (Batalin–Vilkovisky algebra)

Graded commutative algebra \((A, d_A, \cdot)\) endowed with a linear operator \(\Delta^2 = 0, \ d_A \Delta + d_A \Delta = 0\), of order 2:

\[
\Delta(abc) = \Delta(ab)c + \Delta(bc)a + \Delta(ca)b - \Delta(a)bc - \Delta(b)ca - \Delta(c)ab.
\]

Examples: \(H_\bullet(TCFT)\), \(H_\bullet(LX)\) (string topology), Dolbeault complex of Calabi-Yau manifolds, the bar construction \(BA\), etc.

Operadic topological interpretation: \(H_\bullet(fD_2) = BV\).
Theorem (Galvez–Tonks–V.)

The **inhomogeneous** Koszul duality theory provides us with a quasi-free resolution $B\!V_\infty := \Omega B\!V_i \sim B\!V$.

Proof. Problem:

$B\!V \cong T(\cdot, \Delta)/(\text{homogeneous quadratic and cubical relations})$

Solution: Introduce

$$[-, -] := \Delta \circ (- \cdot -) - (\Delta(-) \cdot -) - (- \cdot \Delta(-))$$

a degree 1 Lie bracket $\implies$ new presentation of the operad $B\!V$:

$$B\!V \cong T(\cdot, \Delta, [ , ]) / (\text{inhomogeneous quadratic relations})$$

Application: $B\!V_\infty$-algebras & $\infty$-morphisms.

Corollary: $\text{HTT} \& \text{Ho}(\text{dg } B\!V\text{-alg}) \cong \infty\text{-dg } B\!V\text{-alg} / \sim_h$. 
Applications in Mathematical Physics

**Application:** Lian–Zuckerman conjecture for Topological Vertex Operator Algebra.

**Theorem (Lian–Zuckerman ’93)**

\[ H_{BRST}(TVOA) : BV\text{-algebra}. \]

**Theorem (Lian–Zuckerman conjecture, Galvez–Tonks–V)**

\[ C_{BRST}(TVOA) = TVOA : \text{explicit } BV_\infty\text{-algebra, which lifts the Lian–Zuckerman operations.} \]

**Remarks:**

- Lian–Zuckerman conjecture similar to the Deligne conjecture.
- Conjecture: some converse should be true, i.e. \( BV_\infty \cong TVOA. \)
Theorem (Barannikov–Kontsevich–Manin)

\((A, d, \cdot, \Delta)\) dg BV-algebra satisfying the \(d\Delta\)-lemma

\[
\ker d \cap \ker \Delta \cap (\text{Im}d + \text{Im}\Delta) = \text{Im}(d\Delta) = \text{Im}(\Delta d)
\]

\[\implies H_\bullet(A, d) \text{ carries a Frobenius manifold structure, which extends the transferred commutative product.}\]

**Application:** B-side of the Mirror Symmetry Conjecture.

**Question:** Application of the HTT for \(BV_\infty\)-algebras???

\[
BV^i \cong T^c(\delta) \otimes Com_1^* \circ \text{Lie}^* \leftrightarrow H_\bullet(\overline{M}_{0,n+1}), \text{ so, not yet!}
\]
Topological interpretation: homotopy trivialization of $S^1$

**Conjecture:** [Costello–Kontsevich] $fD_2 / h S^1 \simeq \overline{M}_{0,n+1}$.

**Theorem (Drummond-Cole – V.)**

Minimal model of $BV : T\left(T^c(\delta) \oplus H^{•+1}(\mathcal{M}_{0,n+1})\right) \overset{\sim}{\longrightarrow} BV$.

**Application:** New notion of $BV_\infty$-algebras.

Homotopy trivialization of the circle $\iff$ trivial action of $T^c(\delta)$

$$H_•(\overline{M}_{0,n+1})^i = H^{•+1}(\mathcal{M}_{0,n+1}) \ & \ Koszul \ [Getzler \ '95]$$

**Solution of the conjecture over $\mathbb{Q}$**

$$BV_\infty / h \Delta = \underbrace{T(H^{•+1}(\mathcal{M}_{0,n+1}))}_{\text{homotopy Frobenius manifold}} \overset{\sim}{\longrightarrow} \underbrace{H_•(\overline{M}_{0,n+1})}_{\text{Frobenius manifold}}.$$
HTT for homotopy BV-algebras with $\Delta$ trivialization

[BKM]: $(A, d, \cdot, \Delta)$ dg BV-algebra satisfying the $d\Delta$-lemma $\implies H_\bullet(A, d)$ carries a Frobenius manifold structure.

Theorem (Drummond-Cole – V.)

$(A, d, \cdot, \Delta)$ dg BV-algebra satisfying the Hodge–de Rham condition $\implies H_\bullet(A, d)$ carries a homotopy Frobenius manifold structure, which extends the Frobenius manifold structure and

$$\text{Rect}(H_\bullet(A), d) \sim (A, d, \cdot, \Delta) \text{ in } \text{Ho}(\text{dg BV-alg})$$
De Rham cohomology of Poisson manifolds

**Theorem (Koszul ’85)**

\((M, \pi)\) Poisson manifold \(\implies\) De Rham complex

\((\Omega^\bullet M, d_{DR}, \wedge, \Delta := [i_\pi, d_{DR}]): BV\text{-algebra.}\)

**Theorem (Merkulov ’98)**: \(M\) symplectic manifold satisfying the Hard Lefschetz condition \(\implies H^\bullet_{DR}(M):\) Frobenius manifold.

**Theorem (Dotsenko–Shadrin–V.)**

*For any Poisson manifold* \(M\) \(\implies H^\bullet_{DR}(M):\) homotopy Frobenius manifold, *s.t.*

\[\text{Rect}(H^\bullet_{DR}(M)) \sim (\Omega^\bullet M, d_{DR}, \wedge, \Delta)\] in \(\text{Ho}(dg BV\text{-alg}).\)

**Generalization:** \((M, \pi, E)\) Jacobi manifold (eg contact),

\((\Omega^\bullet M, d_{DR}, \wedge, \Delta_1 := [i_\pi, d_{DR}], \Delta_2 := i_\pi i_E): BV_\infty\text{-algebra.}\)
In many areas of mathematics some "higher operations" are arising. These have become so important that several research projects refer to such expressions. Higher operations form new types of algebras.

The key to understanding and comparing them, to creating invariants of their action is operad theory. This is a point of view that is four oldstyle/zero.oldstyle years old in algebraic topology, but the new trend is its appearance in several other areas, such as algebraic geometry, mathematical physics, differential geometry, and combinatorics.

The present volume is the first comprehensive and systematic approach to algebraic operads. An operad is an algebraic device that serves to study all kinds of algebras (associative, commutative, Lie, Poisson, A-infinity, etc.) from a conceptual point of view.

The book presents this topic with an emphasis on Koszul duality theory. A modern treatment of Koszul duality for associative algebras, the theory is extended to operads. Applications to homotopy algebra are given, for instance the Homotopy Transfer Theorem. Although the necessary notions of algebra are recalled, readers are expected to be familiar with elementary homological algebra. Each chapter ends with a helpful summary and exercises. A full chapter is devoted to examples, and numerous figures are included.

A low-level chapter on Algebra, accessible to (advanced) undergraduate students, the level increases gradually through the book. However, the authors have done their best to make it suitable for graduate students: three appendices review the basic results needed in order to understand the various chapters. Since higher algebra is becoming essential in several research areas like deformation theory, algebraic geometry, representation theory, differential geometry, algebraic combinatorics, and mathematical physics, the book can also be used as a reference work by researchers.

Thank you!
In many areas of mathematics some "higher operations" are arising. These have become so important that several research projects refer to such expressions. Higher operations form new types of algebras. The key to understanding and comparing them, to creating invariants of their action is operad theory. This is a point of view that is four oldstyle/zero.oldstyle years old in algebraic topology, but the new trend is its appearance in several other areas, such as algebraic geometry, mathematical physics, differential geometry, and combinatorics.

The present volume is the first comprehensive and systematic approach to algebraic operads. An operad is an algebraic device that serves to study all kinds of algebras (associative, commutative, Lie, Poisson, A-infinity, etc.) from a conceptual point of view. The book presents this topic with an emphasis on Koszul duality theory. A modern treatment of Koszul duality for associative algebras, the theory is extended to operads. Applications to homotopy algebra are given, for instance the Homotopy Transfer Theorem. Although the necessary notions of algebra are recalled, readers are expected to be familiar with elementary homological algebra. Each chapter ends with a helpful summary and exercises. A full chapter is devoted to examples, and numerous figures are included.

A first low-level chapter on Algebra, accessible to (advanced) undergraduate students, the level increases gradually through the book. However, the authors have done their best to make it suitable for graduate students: three appendices review the basic results needed in order to understand the various chapters. Since higher algebra is becoming essential in several research areas like deformation theory, algebraic geometry, representation theory, differential geometry, algebraic combinatorics, and mathematical physics, the book can also be used as a reference work by researchers.

Thank you!